PAK Project – Course Material

PAK101 – Welcome. The Lambert W-function analytical approach

 8 Jan 2019 ver 0v5

Level 1 – How to solve basic diode an transistor circuits

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PAK101 – Welcome. The Lambert W-function analytical approach

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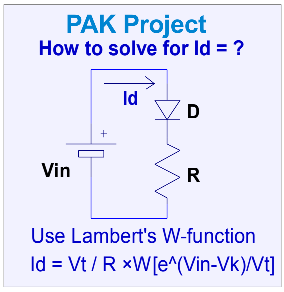
Engineers are coy when it comes to analysing transistor circuits. We can do small signal analysis for transistor circuits. But we cannot do large-signal analysis for transistor circuits. Why? Because it is not taught and it's not in the text books. That's because transistors give “transcendental” equations which means we didn't until recently have any standard functions like log, tan, to solve transistor circuits. We had to use simulators. But since 2000, we can write solutions for the simple common emitter transistor amplifier circuit using the new Lambert W-function.

A resistor in series with a diode can now be solved analytically for large-signal thanks to Robert Banwell See the equation in the diode plus resistor circuit (Right). The W(…) is the Lambert-W function. 'Vk' is a knee voltage x-axis intercept similar to a diodes turn-on voltage.



Robert Banwell's diode solution allows a large signal analytical solution for the common emitter amplifier (see IEEE paper in T.CAS November 2000). This is because BJT's are modelled as diodes in all the SPICE models.

SPICE simulators have been around for nearly 50 years but still uses the same internal numerical solver engine.
SPICE transistor models have improved slowly. The latest VBIC model [ref] is complex!
Simulators hide most of the complexity of transistor circuits and models from the user.
Simulators are a blunt tool for circuit designers because they don't allow large signal *circuit analysis*! For the best circuit designs we need general large signal *circuit analysis*.

The advantages of an analytic solution:

- Describes general behaviour, as opposed to a numerical result that is based on specific values and initial conditions.
- Can contribute to intuitive understanding as parameters change.
- Can be differentiated and integrated for special conditions such as maxima's & minima's.
- Has no convergence problems.
- Can provide an initial DC operating point values for AC simulations.
- Can provide analytical device models for BJT's, MOSFET's and tubes to enhance simulators.
- Can provide faster simulations – useful for digital effects units and Real Time Emulators (RTE's) which need to run 2 to 3 *orders-of-magnitude* faster than most SPICE simulators.

The problem with the analytic solution approach is that you can only analyse transistor circuits that have been already solved. There is still no known way to solve capacitors in transistor circuits (for the large-signal case), so we have to use numerical methods for AC simulations for a while yet.

Robert Banwell only solved one basic transistor circuit – the single transistor Common Emitter amplifier with shunt biasing and no beta-fall, Early effect or Vce saturation. So we need to find a solution with these secondary effects (see PAK207-9). We also need solutions for transistor circuits for power amplifiers – most of these are hard to solve.

Once we have solved most of the standard transistor circuit building blocks we can use them in SPICE simulators for better models, faster simulations and reduce the chances of non-convergence (now wouldn't that be nice!)

Next, PAK102 explains the Lambert W diode equation and it's derivation.

PAK Project – Course Material

PAK102 – Lambert W-function analytical solution for the D+R

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The most basic circuit that was said "cannot be solved" is the diode plus resistor.
The diode plus resistor solution is needed for solving transistor circuits.

The solution for a diode plus a resistor (D+R)

Figure 1 shows the circuit of an ideal diode with a series resistor and fed by a voltage source. The solution is

$$I_D = \frac{Vt}{R} \cdot W\left[e^{\frac{Vin - Vk}{Vt}}\right]$$

where Vk is the "knee voltage" [my terminology] given by

$$Vk = Vt \cdot \ln\left(\frac{Vt}{Is \cdot Rs}\right) - Is \cdot Rs$$

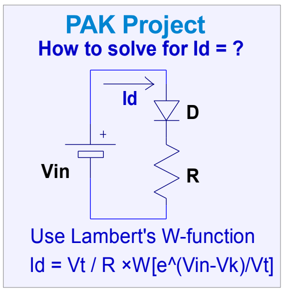


Fig.102a Basic D+R

Now let's unpack what's going on in these equations.

First note that usually a diodes saturation current '*Is*' is very very small (fA to pA) so *Is*×*Rs* voltage is in the pV to nV range. This means the Ln(Vt/(*Is*·*Rs*)) term dominates over the *Is*·*Rs* term in the Vk equation.

Vk determines how much input voltage is needed to get significant current through the diode and resistor so it is not just the diode that determines the current flow by *both* the diode and the resistor. It is like the threshold voltage in MOSFET's. Significant conduction happens when Vin approaches Vthres. For a diode, when Vin is well above Vk the diode is effectively acting as a constant voltage drop (~Vk volts) and any additional voltage is dropped across the resistor.

The Vk equation is effectively the number of *decades* times that *Is*·*Rs* is *relative* to Vt a thermally generated diode internal reference voltage. If you changed the Ln(…) to Log(…) then the Vt in front of the Ln(…) gets scaled by 2.3×Vt so Vk changes by typically 60mV per decade for Is·Rs relative to Vt. Typically with Is in the sub nA region and I in the mA or amp region Vk needs to be 1e6 to 1e9 or 6 to 9 decades or 14 to 20 nepers times one Vt. So with 6 to 9 decades at 60mV/decade the Vk voltage is in the range of 360mV to 540mV.

Since Vk is in the exponent with Vin the knee voltage Vk is effectively a scale factor k for the $e^{\wedge Vin/Vt}$ part in the W[k×$e^{\wedge Vin/Vt}$] function. More on the nature of the W-function later. The W-function returns a non-dimensional number just like Ln(…) and Exp(…) do. Also, the W-function must take a non-dimensional number too, just like Ln(…) and Exp(…) do.

After the W[k×$e^{\wedge Vin/Vt}$] function the input current is calculated as I=Vt/R·W[k×$e^{\wedge Vin/Vt}$] which is simply Ohms Law I=V/R, where V is voltage developed across the resistor and V=Vt·W[k×$e^{\wedge Vin/Vt}$], again, where the W(…) function is a scaler number with no dimensions.

That gives some intuitive insights into what's going on in the Lambert's W diode plus resistor equation solution.

Next, the actual steps for solving the Shockley diode equation with a series resistor.

But first a refresher of a few basic identities for solving equations using Lambert's W function.

Often used exponential and log identities:

- $e^a \cdot e^b = e^{(a+b)}$
- $\left(e^a\right)^2 = e^{2a} \qquad \frac{1}{\left(e^a\right)} = e^{-a}$
- $-\ln(a) = \ln\left(a^{-1}\right) = \ln\left(\frac{1}{a}\right)$

- $\ln(a \cdot b) = \ln(a) + \ln(b)$
- $\ln(a+b)$? Be careful! This is easily confused with the above.

D+R equation solving steps

The following explains all the steps in detail (my apology to those who know all this).

The equation for the current through an ideal diode is given by the Shockley diode equation

$$I_D = Is \cdot \left(e^{\frac{V_D}{Vt}} - 1 \right) \tag{1}$$

where *Vt* is the 'thermal voltage' given by Boltzman's constant k_B, absolute temperature *T*, and q_e is the electron charge. The term *Vt* is

$$Vt = \frac{k_B T}{q_e} \quad \text{or 25.86mV at 27°C ~300K°} \tag{2}$$

Using the voltage loop method we write

$$Vin = I_D \cdot Rs + V_D \tag{3}$$

We eliminate V_D in Equation (1) using

$$V_D = Vin - I_D \cdot Rs \tag{4}$$

This gives

$$I_D = Is \cdot \left[e^{\left(\frac{Vin - I_D \cdot Rs}{Vt} \right)} - 1 \right] \tag{5}$$

We want to solve Equation (5) for I_D in terms of the *Vin*. The Lambert W-function is a relatively new function on the block that allows a solution for Id for the diode plus resistor. The Lambert W function is used to solve for y in the expression

$$y \cdot e^y = e^x \quad \text{then we write} \quad y = W\left(e^x\right) \tag{6}$$

To explain this using the following expression

$$Y^2 = X \quad \text{the inverse is written as} \quad Y = \pm\sqrt{X} \tag{7}$$

When we apply the inverse function to Y^2 =X we take the square-root of both sides – reducing the LHS Y^2 to Y, and ±√X on the RHS. Then we can use the square-root algorithm to get a numeric value ±√X and then we have the numerical answer for Y.

We use the same process for the Lambert W-function to both sides – to reduce the y·e^y part to just y, and the RHS becomes W(e^x). Then we can use a Lambert W algorithm to get W(e^x) and we have a numerical answer for y. Got it?

Algorithms for W(e^x)

An online link for W(*X*) of up to 99 sig. Figs. is here
http://functions.wolfram.com/webMathematica/FunctionEvaluation.jsp?name=ProductLog

A useful approximation for W(e^x) to about 2% worst case accuracy is [ref: Miranda,Ortiz 2006].

$$W\left(e^x\right) \simeq \ln\left(1+e^x\right)\left(1 - \frac{\ln\left(1+\ln\left(1+e^x\right)\right)}{2+\ln\left(1+e^x\right)}\right) \tag{8}$$

The accuracy can be improved to 0.04% using a second step – an LTspice listing is given below where We2(X) is double precision and We1(X) is single precision:

```
.function We2(X) {If(U(X+30),We1(X-Ln(1+We1(X)+Ln(We1(X))-X)),We1(X))}
```

```
.function We1(X) {(Le(X)*(1-(Ln(1+Le(X)))/(2+Le(X))))} ;Unlimited W(e^X)) 2%
.function Le(X) {If(U(X-450), X, Ln(1+Exp(X)))} ;Unlimited Ln(1+e^X)
```

Note the 'e' in 'We' means x is raised to the power of e.
Le(X) is the SpoftPlus (soft Max) which is conditional to handle X large values without crashing (in excess of 450 meaning e^450 or ~10^200) which is seems large but it is easily reached with diodes with only 12 volts input (since 450×0.026V=11.7V).
Alternatively Newton's method can be used for higher precision [ref Jin He 2007].

Back to solving Equation (5) $I_D = Is \cdot \left[e^{\left(\frac{Vin - I_D \cdot Rs}{Vt}\right)} - 1 \right]$.

We have to do some algebra to rearrange it into the form of $y \cdot e^y = e^x$ which is of the form

$$f(I_D) \cdot e^{f(I_D)} = e^{f(Vin)} \tag{9}$$

Note the $f(I_D)$ term ahead of the 'e' must be exactly the same as the exponent $f(I_D)$ term to allow a solution using W(*x*). I explain the reason for this using the quadratic equation solution.

Sidebar explanation
Recall the method used to solve a general quadratic equation is by *completing of the square*. [http://www.purplemath.com/modules/sqrquad.htm & http://en.wikipedia.org/wiki/Quadratic_equation]. This is because we can only take an inverse of $Y^2 = X$ to write the solution for y as $\pm\sqrt{X} = Y$.
We can't solve for y in $a\,y^2 + by = x$ directly, we have to get into the form $(Y)^2 = X$ as follows: $a\,y^2 + by + c = 0$ => $y^2 + \frac{b}{a}y + \frac{c}{a} = 0$ >complete the square> $\left(y + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$

$\left(y + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$ >now apply inverse> $\left(y + \frac{b}{2a}\right) = \pm\sqrt{\left(\left(\frac{b}{2a}\right)^2 - \frac{c}{a}\right)}$

$y + \frac{b}{2a} = \pm\sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$ now unpack to get $y = \frac{-b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$

Now the steps to solve our diode plus resistor equation:

Step1. Equation (5)

$$I_D = Is \left[e^{\left(\frac{Vin - I_D \cdot Rs}{Vt}\right)} - 1 \right] \tag{10}$$

Move *Is* to LHS

$$I_D + Is = Is \cdot e^{\left(\frac{Vin - I_D \cdot Rs}{Vt}\right)} \tag{11}$$

Use the identity $e^{(a+b)} = e^a \cdot e^b$

$$\left(I_D + Is\right) = Is \cdot e^{\left(\frac{Vin}{Vt}\right)} e^{\left(\frac{-I_D \cdot Rs}{Vt}\right)} \tag{12}$$

(Effectively, to move the *Id* term to LHS you can swap sides and swap sign sign for exponents)

$$\left(I_D + Is\right) e^{\left(\frac{I_D \cdot Rs}{Vt}\right)} = Is \cdot e^{\left(\frac{Vin}{Vt}\right)} \tag{13}$$

We are working toward balancing the LHS exponent and LHS mantissa ***fn e***$^{\boldsymbol{fn}}$
Multiplying both sides by (Rs/Vt)

$$\left(I_D+Is\right)\left(\frac{Rs}{Vt}\right)e^{\left(\frac{I_D\cdot Rs}{Vt}\right)}=Is\left(\frac{Rs}{Vt}\right)e^{\left(\frac{Vin}{Vt}\right)} \tag{14}$$

We can 'add' an *Is* terms to the LHS <u>exponent</u> to get and (I_D+Is) in the <u>exponent</u>. We can do this by multiplying both sides by a constant of the form e^($^{Is\cdot Rs}/_{Vt}$) like this:

$$\left(I_D+Is\right)\left(\frac{Rs}{Vt}\right)e^{\left(\frac{I_D\cdot Rs}{Vt}\right)}e^{\left(\frac{Is\cdot Rs}{Vt}\right)}=Is\left(\frac{Rs}{Vt}\right)e^{\left(\frac{Vin}{Vt}\right)}e^{\left(\frac{Is\cdot Rs}{Vt}\right)}\quad =>$$
$$\frac{\left(I_D+Is\right)\cdot Rs}{Vt}e^{\left(\frac{\left(I_D+Is\right)\cdot Rs}{Vt}\right)}=\frac{Is\,Rs}{Vt}e^{\left(\frac{Vin+Is\cdot Rs}{Vt}\right)} \tag{15}$$

Looking good. We now have it in the form ***fn*** *e*^***fn*** ready to use the Lambert's W-function.

Step 2. We have the same exponent and mantissa on the LHS, it's in the form ***fn*** *e*^***fn*** where

$$fn=\frac{\left(I_D+Is\right)\cdot Rs}{Vt} \tag{16}$$

So we can now apply the Lambert W inverse function W(*x*) to both sides.
For the LHS the inverse of ***fn*** *e*^***fn*** is "*fn*"
For the RHS we get W(*RHS*)
Put together we get $fn=\frac{\left(I_D+Is\right)\cdot Rs}{Vt}=W\left(RHS\right)$ which is

$$\frac{\left(I_D+Is\right)\cdot Rs}{Vt}=W\left(\frac{Is\,Rs}{Vt}\cdot e^{\left(\frac{Vin+Is\cdot Rs}{Vt}\right)}\right) \tag{17}$$

To get I_D by itself unpack it to solve for I_D giving

$$I_D=\frac{Vt}{Rs}\cdot W\left(\frac{Is\,Rs}{Vt}\cdot e^{\left(\frac{Vin+Is\cdot Rs}{Vt}\right)}\right)-Is \tag{18}$$

Step 3. We can simplify the RHS. Appendix 102 shows the steps for this.

$$I_D=\frac{Vt}{Rs}\cdot W\left[e^{\left(\frac{Vin-Vk}{Vt}\right)}\right]-Is\quad\text{where}\quad Vk=Vt\cdot\ln\left(\frac{Vt}{Is\cdot Rs}\right)-Is\cdot Rs \tag{19}$$

The diode plus resistor *Vk* voltage is the *x*-axis intercept on a linear-linear plot of *Id* versus *Vin* for the diode plus resistor circuit plus about 60mV or about 2·Vt volts. (see plot below)

Note: The solution above omitted the diode non-ideality factor '*n*' or NF in the SPICE GP model and EM models. If you want to include a non-ideal '*n*' for Vt then add it to every "Vt" in these solutions.

Summary: For the original diode plus resistor equation and solution are:

$$I_D=Is\cdot\left[e^{\left(\frac{Vin-I_D\cdot Rs}{Vt}\right)}-1\right]\quad =>$$
$$I_D=\frac{Vt}{Rs}\cdot W\left[e^{\left(\frac{Vin-Vk}{Vt}\right)}\right]-Is\quad\text{where}\quad Vk=Vt\cdot\ln\left(\frac{Vt}{Is\cdot Rs}\right)-Is\cdot Rs \tag{20}$$

21(20)

Validation by simulation

Figure 102b. shows the plot for a diode with a 1 ohm series resistance and Is=10nA. The equation solution plot are not visible unless they are plotted slightly above (10mV). The Vk is 501mV (read as V(vp) plot) but the *x*-axis intercept is about 560mV, or about 60mV higher than the Vk value by about 2Vt.

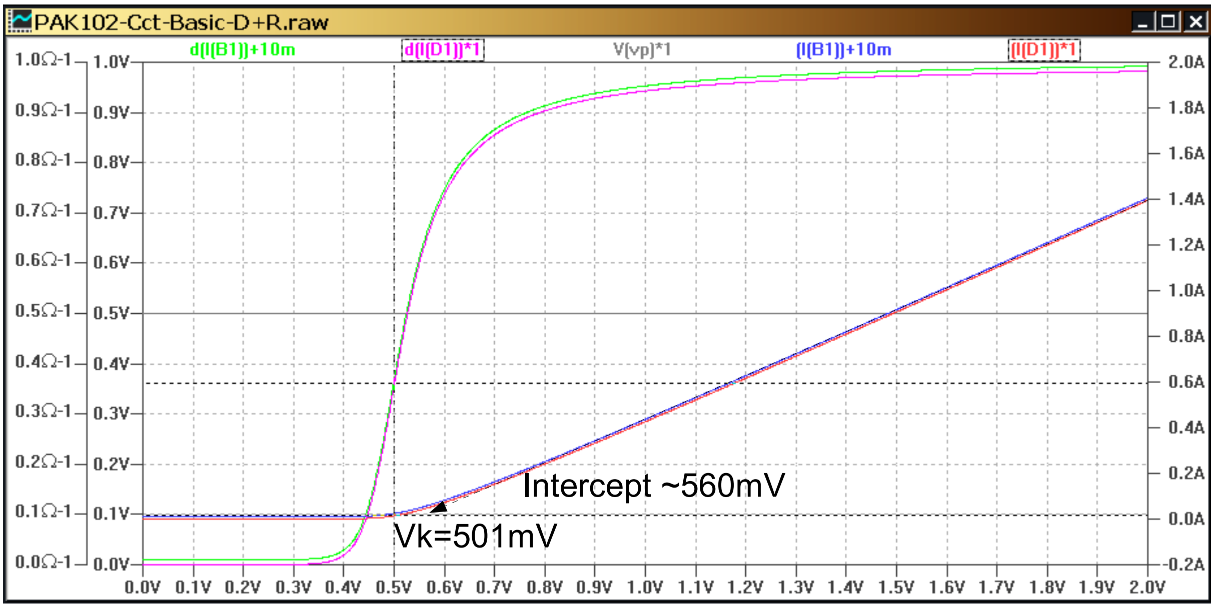
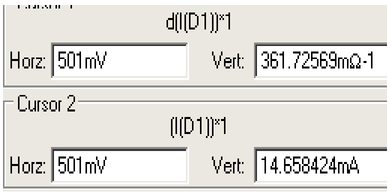


Figure 102b. Plots for a diode with a 1Ω series resistance.



Cursor readout Id & d(Id) at Vk 501mV

The cursor readout for d(Id) shows as 361.7mA/V or 36.17% of the maximum value due to using 1 ohm in this example. This is relevant to setting optimum bias for Class-AB amplifiers which is covered in PAK301.

What have we achieved by solving the current in this simple circuit?

- We have an exact solution. It is a general solution – special version of Ohm's Law for a diode and a resistor. We don't have to use a simulator to find the current given Vin.
- We can now use this D+R solution to solve basic transistor circuits, like the common emitter amplifier!

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Appendix for PAK102

Some useful log and exponential rearrangements:

- $\ln(a+b) = \ln\left(a\cdot\left(1+\frac{b}{a}\right)\right) = \ln(a)+\ln\left(1+\frac{b}{a}\right)$
- $e^a+e^{-a} = e^a\cdot\left(1+e^{-2a}\right) = e^a\cdot e^{\ln\left(1+e^{-2a}\right)} = e^{a+\ln\left(1+e^{-2a}\right)}$ is useful for a≥0 to make e^a the dominant part growing, and Ln(...) term diminishing. Note the Ln(1+e^x) is the SoftPlus or soft Max/Min function frequently used in modelling, neural nets etc. The Ln(Cosh(x)) = Ln(1/2(e^x+e^-x)) function gives a soft Abs(x) function.
- When a≤0 use $e^a+e^{-a} = e^{-a}\cdot\left(1+e^{2a}\right) = e^{-a}\cdot e^{\ln\left(1+e^{2a}\right)} = e^{-a+\ln\left(1+e^{2a}\right)}$ for a dominant

e^a component.

- Note for $e^{a}-e^{-a} = e^{a}\cdot\left(1-e^{-2a}\right) = e^{a}\cdot e^{\ln\left(1-e^{-2a}\right)} = e^{a+\ln\left(1-e^{-2a}\right)}$ needs a>0 to avoid a Ln(0)→−∞ overflow error. When a=0 we have $e^{a}-e^{-a} = 0$.
  Tip: For numerical continuity as |a| approaches 0 use $e^{a}-e^{-a} = e^{a}\cdot\left(1-e^{-2a}\right)$ then the e^a component is still the dominant term. These are exact, no approximations.

Derivation of Vk knee offset voltage

For circuit solutions it is more convenient to write

$$I_D = \frac{Vt}{Rs}\cdot W\left(\frac{Is\cdot Rs}{Vt}\cdot e^{\left(\frac{Vin+Is\cdot Rs}{Vt}\right)}\right) - Is \tag{21}$$

as

$$I_D=\frac{Vt}{Rs}\cdot W\left[e^{\left(\frac{Vin-Vk}{Vt}\right)}\right] - Is \quad \text{where} \quad Vk=Vt\cdot\ln\left(\frac{Vt}{Is\cdot Rs}\right) - Is\cdot Rs \tag{22}$$

Derivation:

The steps to derive Vk are

$$I_D = \frac{n\cdot Vt}{Rs}\cdot W\left[e^{\ln\left(Is\frac{Rs}{n\cdot Vt}\right)}\mathrm{e}^{\left(\frac{(Vin+Is\cdot Rs)}{n\cdot Vt}\right)}\right] - Is \quad =>$$

$$I_D = \frac{n\cdot Vt}{Rs}\cdot W\left[e^{\frac{(Vin+Is\cdot Rs)}{n\cdot Vt}+\ln\left(\frac{Is\cdot Rs}{n\cdot Vt}\right)}\right] - Is \quad =>$$

$$I_D = \frac{n\cdot Vt}{Rs}\cdot W\left[e^{\frac{(Vin+Is\cdot Rs)}{n\cdot Vt}+\frac{n\cdot Vt\ln\left(\frac{Is\cdot Rs}{n\cdot Vt}\right)}{n\cdot Vt}}\right] - Is \quad =>$$

$$I_D = \frac{n\cdot Vt}{Rs}\cdot W\left[e^{\frac{(Vin+Is\cdot Rs)}{n\cdot Vt}+\frac{n\cdot Vt\ln\left(\frac{Is\cdot Rs}{n\cdot Vt}\right)}{n\cdot Vt}}\right] - Is \quad =>$$

$$I_D = \frac{n\cdot Vt}{Rs}\cdot W\left[e^{\frac{Vin+Is\cdot Rs+n\cdot Vt\ln\left(\frac{Is\cdot Rs}{n\cdot Vt}\right)}{n\cdot Vt}}\right] - Is \quad =>$$

$$I_D = \frac{n\cdot Vt}{Rs}\cdot W\left[e^{\left(\frac{Vin+Is\cdot Rs-n\cdot Vt\cdot\ln\left(\frac{n\cdot Vt}{Is\cdot Rs}\right)}{n\cdot Vt}\right)}\right] - Is$$

finally $I_D=\frac{Vt}{Rs}\cdot W\left[e^{\left(\frac{Vin-Vk}{Vt}\right)}\right] - Is$ where $Vk = n\cdot Vt\cdot\ln\left(\frac{n\cdot Vt}{Is\cdot Rs}\right) - Is\cdot Rs$

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PAK Project – Course Material

PAK103 – Basic BJT analytical solution using the D+R solution

 8 Jan 2019 ver 0v5

Solving a basic BJT amp

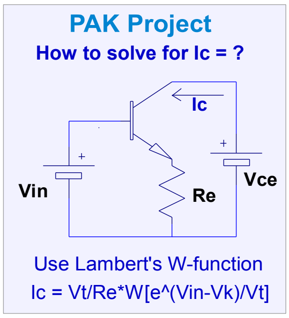

Fig103a Basic BJT amp

The Ebers & Moll (EM) BJT model was developed around 1954 and it was enhanced by Gummel & Poon (GP) in 1970, and the VBIC model 1995.

All spice SPICCE BJT models use the base to emitter voltage (Vbe) to generate the collector current in the normal *active region* (which is when the collector-base junction is reverse biased and the base-emitter junction is forward biased).

In SPICE BJT models collector current is related to the base to emitter voltage (Vbe) by a modified Schockley diode equation provided Vce>Vbe then

$$Ic = Is \cdot \left(\exp\left(\frac{Vbe}{Vt} \right) - 1 \right) \tag{23}$$

where *Is* is the collector-side saturation current and *Vt* is thermal voltage Equation (2) *Vt = KT/q*.

The Vbe used here is the effective junction potential after ohmic voltage drops are subtracted from the transistors terminal voltages. All SPICE models place these ohmic resistances for the base, emitter and collector outside the model equations; this was necessary because they could not be solved at the time the models were formulated because they didn't have Lambert's W-function. Instead, the SPICE numerical engine solves these resistances as if they were added resistances in your circuit. PAK solutions can now solve these resistances in the new Lambert W-model equations. See below.

The collector side saturation current Is was chosen by GP and EM because it makes it easier to formulate the base current by generating the collector current first, where Ic is does not have any beta-fall effects included at this stage. This means the Is in the Ic equation is a factor of Beta times greater than the Is that is measured from the actual base-emitter-diode voltage and base current.

The revised VBIC model use very different physical mechanisms control the collector and base currents. VBIC explicitly separates the base and collector current modelling because "collector current primarily depends on the base doping, and the base current depends primarily on recombination and generation in the emitter region [ref McAndrew Bctm98VbicText 1994].

The GP and EM approach the base current is generated from collector current (Ic) divided by the current gain (Beta) as a 'backwards' gain. So the equation for the base current is:

$$Ib = \left(\frac{Is}{Beta} \right) \cdot \exp\left(\frac{Vbe}{Vt} \right) \tag{24}$$

Interestingly, the SPICE BJT equations model transistors as transconductance devices in other words as voltage controlled current sources. BJT's are not modelled as a current controlled device, as is commonly taught in lower level courses using the water tap analogy. You can design transistor circuits by thinking of them as current controlled devices, but that doesn't prove that they *work* by current control.

Putting it another way, SPICE models the BJT transistor effectively is a *field-effect* transistor where base current gives rise to the base voltage field that then generate the collector current. Bipolar transistors and FET's are obviously constructed very differently, but for their SPICE models they are the same cause-and-effect relationship – they are both V-to-I or transconductors.

Interestingly, recent developments have made *junction-less* FET's possible that do not need p and n materials [ref Colinge 2010]. These FET's exhibit near-ideal subthreshold slope, extremely low leakage currents, and less degradation of mobility with gate voltage and temperature than classical FET's. Even energy gap-less FET's have been made using graphine [ref Jang PNAS 2013].

Next, we use the diode plus resistor solution to solve the common emitter amplifier.

Common Emitter BJT with emitter degeneration

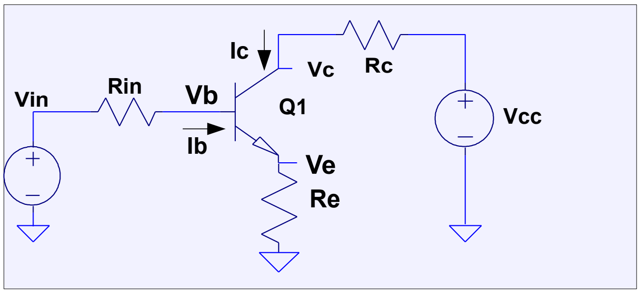

Fig.103a. Basic CE with emitter degeneration

Figure 103a shows a Single Ended BJT Common Emitter amplifier with an emitter degeneration resistor Re and a base resistor Rin and a collector load resistor Rc.

Rin could be a base stopper resistor or it can represent a lumped Thévenin equivalent resistance for a signal generator, etc. Re can be a parasitic (internal terminal) resistance or may be an external resistor such as an emitter degeneration resistor.

Figure 103b shows a BJT modelled by a diode with a series base resistance *Rin* and an external emitter resistance *Re*.

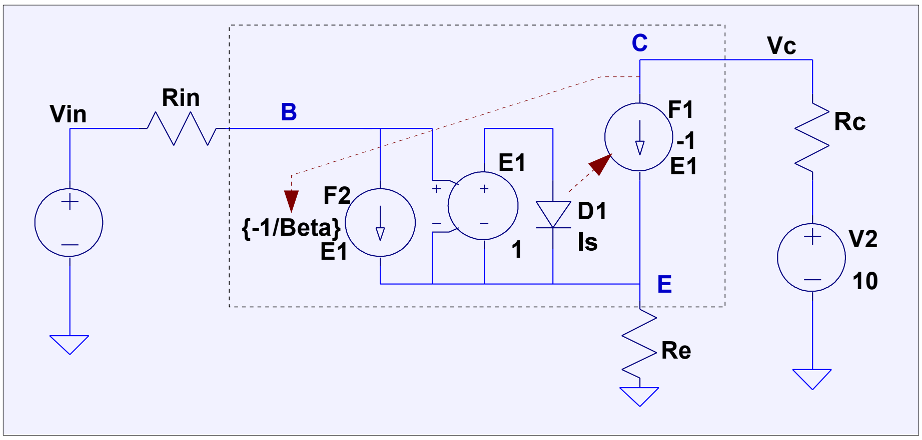

Fig.103b. Basic CE with collector-side diode and Ib = Ic / Beta

In **Figure 103b** the base current does not flow through a diode as such. Instead the base current is generated by a current source that gives Ib = Ic / Beta. The collector current is generated by a buffered diode D1 with Is saturation current. This follows the traditional GP and EM model equations for a transistor in saturation. The reverse condition is not modelled here since it is seldom encountered in amplifier circuits.

The previous diode plus resistor solution can be applied to **Figure 103b** to solve this common emitter amplifier with emitter degeneration. We need to refer the base resistors back to the emitter side to give an equivalent resistance that acts in series with the collector-side diode using

$$Req = \frac{Rin}{Beta} + Re\left(\frac{1+Beta}{Beta}\right) \tag{25}$$

Why is there a factor of (Beta+1)/Beta applied to Re? It's because *Re* carries emitter current, not collector current, and emitter current gives more voltage drop than collector current due to the extra base current component. This can bee seen from the following equations

$$Ie = Ic + Ib \quad \text{so} \quad Ie = Ic + \frac{Ic}{Beta} = Ic\left(1 + \frac{1}{Beta}\right) \quad \text{or} \quad Ie = Ic\left(\frac{Beta+1}{Beta}\right) \tag{26}$$

Applying our previous diode plus resistor solution (PAK102) we can write Ic as

$$Ic = \frac{Vt}{Req} \cdot W\left[e^{\frac{Vin-Vk}{Vt}}\right] - Is \quad \text{Where}$$

$$Req = \frac{Rin}{Beta} + Re \cdot \left(\frac{1+Beta}{Beta}\right) \quad \& \quad Vk = Vt \cdot \ln\left(\frac{Vt}{Is \cdot Req}\right) - Is \cdot Req \tag{27}$$

where 'Is' is the SPICE 'IS' parameter . The W[...] can be calculated using Equations given in PAK102 but copied below for LTspice; We2(X) is for double precision (0.04% max error) and We1(X) is single precision (2% max error):

```
.function We2(X) {If(U(X+30),We1(X-Ln(1+We1(X)+Ln(We1(X))-X)),We1(X))}
.function We1(X) {(Le(X)*(1-(Ln(1+Le(X)))/(2+Le(X))))} ;Unlimited W(e^X)) 2%
.function Le(X) {If(U(X-450), X, Ln(1+Exp(X)))} ;Unlimited Ln(1+e^X)
```

Note the 'e' in 'We' means x is raised to the power of e, so there's no need to apply the e first.

Summary:

We have succeeded in solving a simple BJT amplifier that includes emitter degeneration and a some base resistance. We have an equation for the large signal collector current and from this we can calculate all the other circuit voltages and currents. Nice!

We can even use Re to make a Voltage Follower (Common Collector stage) by taking the output from the emitter. Cool!

For the CE solution the output voltage Vc is calculated using Ic from the above equation with

$$Vout = Vcc - Ic \cdot Rc \tag{28}$$

Figure 103c shows simulations and the equations for Ic and output voltages Vc for the CE case. They are virtually identical. The only visible difference is the Vce undershoots and does not stop due to base-collector saturation. PAK209 adds Vce saturation.
Plot key: LTspice Pink for Vout. Vcc=10V, Beta=100 , Rin=1k, Rc=1k, Re step 1mR,10R,100R.

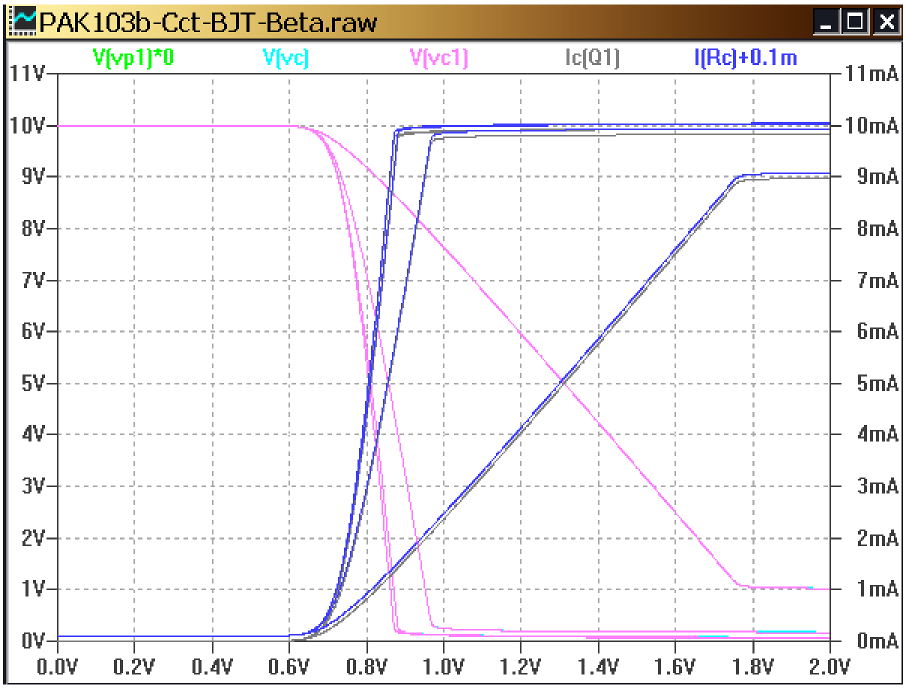

Figure 103c. D+R solution over-plotted with LTspice BJT IRc offset 0.1mA. This shows the equations are correct

In **Fig 103c** the saturation has been added by placing a diode from base to collector with the *same Is* as D1 in Fig 103b. This diode conducts when Vce falls below Vt·Ln(Beta). The overall model is excellent but does not include the Early effect or Beata fall at high currents or low currents (covered in PAK207).

PAK Project – Course Material

PAK104 – Banwell's CE solution with shunt feedback

Banwell's CE solution with shunt voltage feedback

In this circuit a feedback resistor is added from collector to base for a Common Emitter amplifier as shown in **Figure 104a. [4.2]** The following solution is taken from Thomas Banwell [ref: Banwell 2000, Fig.12 Eq.21].

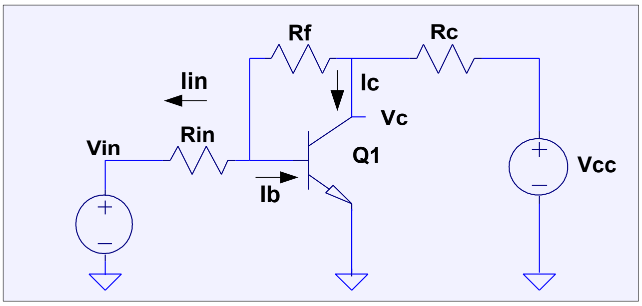

Fig.104a. Banwell's CE with shunt feedback

Three equations are written. First the input voltage to the base node: For *Rin* write

$$Vbe - Vin = Iin \cdot Rin \quad \text{where} \qquad 37(29)$$

Note *Iin* is defined as flowing from the power source through *Rc* and *Rf*. Although voltage loops can be defined arbitrarily, this an intuitive approach (based on the power source). Repeating this for the voltage across *Rf* due to current *Iin* plus *Ib :*

$$\text{for } Rf \ldots \quad V_{BC} = Vc - Vbe = Iin \cdot Rf + \frac{Ic}{Beta} \cdot Rf \qquad 38(30)$$

$$\text{for } Rc \ldots \quad V_{Rc} = Vcc - Vc = Iin \cdot Rc + Ic \cdot Rc + \frac{Ic}{Beta} \cdot Rc \qquad 39(31)$$

We first eliminate *Vc* using (38) and (39) then eliminate *Iin* using (37). Rearrange (38) and (39)

$$Vc = Vbe - Iin \cdot Rf - \frac{Ic}{Beta} \cdot Rf \quad \text{and} \quad Vc = Vcc - Iin \cdot Rc - Ic \cdot Rc - \frac{Ic}{Beta} \cdot Rc$$

$$\text{equating} \quad Vbe - Iin \cdot Rf - \frac{Ic}{Beta} \cdot Rf = Vcc - Iin \cdot Rc - Ic \cdot Rc - \frac{Ic}{Beta} \cdot Rc$$

$$\text{collecting terms} \ldots \quad Ic\left(Rc + \frac{Rf + Rc}{Beta}\right) = Vcc - Vbe - Iin(Rf + Rc)$$

Substitute $Rc' = Rc + (Rf + Rc)/Beta$. Now eliminate *Iin*

$$Ic\,Rc' = Vcc - (Vbe + Ve) - \left(\frac{Rf + Rc}{Rin}\right)(Vbe - Vin)$$

We can now collect *Ve* into *Ic*. First get *Ve* to LHS

$$Ic\,Rc' + Ve\left(1 + \frac{Rf + Rc}{Rin}\right) = Vcc - \left(1 + \frac{Rf + Rc}{Rin}\right)Vbe - \left(\frac{Rf + Rc}{Rin}\right)Vin$$

then substitute *Ve* and substitute $R_{TOT} = Rin + Rf + Rc$

$$Ic\,Rc' = Vcc - \left(\frac{R_{Tot}}{Rin}\right)Vbe - \left(\frac{Rf + Rc}{Rin}\right)Vin \quad =>$$

$$Ic \cdot Rc' = Vcc - \left(\frac{R_{Tot}}{Rin}\right)Vbe - \left(\frac{R_{TOT}}{Rin}\right)Vin \quad \text{simplify}$$

$$Ic\, Rc' = Vcc - \left(\frac{R_{TOT}}{Rin}\right)Vbe - \left(\frac{R_{TOT}}{Rin}\right)Vin \quad \text{where} \qquad 40(32)$$

Next, *Vbe* can be eliminated using the log form of the Shockley diode Equation, giving

$$Ic\, Rc' = Vcc - \left(\frac{R_{TOT}}{Rin}\right)Vt \cdot \ln\left(\frac{Ic}{Is}+1\right) - \left(\frac{Rf+Rc}{Rin}\right)Vin \qquad 41(33)$$

We now proceed to get the above equation into the form of *fn* $e^{\wedge fn} = fn(e^{\wedge Vin})$ where *Ic* terms appear on the LHS, and *Vin* terms appear on the RHS.

Moving terms in front of the log gives

$$\ln\left(\frac{Ic}{Is}+1\right) + Ic\, Rc'\left(\frac{Rin}{R_{TOT}\, Vt}\right) = -\left(\frac{Rin}{R_{TOT}\ Vt}\right)\left(\left(\frac{Rf+Rc}{Rin}\right)Vin + Vcc\right) \quad => \qquad (34)$$

$$\ln\left(\frac{Ic+Is}{Is}\right) + Ic\,\frac{Rc''\, Rin}{R_{TOT}\, n \cdot Vt} = -\left(\left(\frac{Rf+Rc}{R_{TOT}\ Vt}\right)Vin + \left(\frac{Rin}{R_{TOT}\ Vt}\right)Vcc\right) \qquad (35)$$

Convert to exponential form (exponentiate both sides)

$$\left(\frac{Ic+Is}{Is}\right)\cdot \exp\left[Ic\,\frac{Rc'\, Rin}{R_{TOT}\, Vt}\right] = \exp\left[-\left(\left(\frac{Rf+Rc}{R_{TOT}\, Vt}\right)Vin + \left(\frac{Rin}{R_{TOT}\, Vt}\right)Vcc\right)\right] \quad =>$$

$$\left(Ic+Is\right)\cdot \exp\left[Ic\,\frac{Req}{Vt}\right] = Is \cdot \exp\left[-\left(\left(\frac{Rf+Rc}{R_{TOT}\, Vt}\right)Vin + \left(\frac{Rin}{R_{TOT}\, Vt}\right)Vcc\right)\right] \qquad (36)$$

Add an *Is* to the exponent (multiply both sides by exp[Is·Req / Vt])

$$\left(Ic+Is\right)\cdot e^{\left((Ic+Is)\frac{Req}{Vt}\right)} = Is \cdot e^{-\left(\left(\frac{Rf+Rc}{R_{TOT}\, Vt}\right)Vin + \left(\frac{Rin}{R_{TOT}\, Vt}\right)Vcc + \frac{Is\, Req}{Vt}\right)} \quad \text{where}$$

$$Req = \frac{Rin}{R_{Tot}}\left(Rc\left(\frac{Beta+1}{Beta}\right) + \frac{Rf}{Beta}\right) + R\,e\left(\frac{Beta+1}{Beta}\right) \qquad (37)$$

Balance the mantissa by multiplying both sides by *Req / Vt*

$$\left(Ic+Is\right)\frac{Req}{n \cdot Vt} \cdot e^{\left((Ic+Is)\frac{Req}{Vt}\right)} = \frac{Is\, Req}{n \cdot Vt} \cdot e^{-\left(\left(\frac{Rf+Rc}{R_{TOT}\, Vt}\right)Vin + \left(\frac{Rin}{R_{TOT}\, Vt}\right)Vcc + \frac{Is\, Req}{Vt}\right)} \qquad (38)$$

We have balanced the LHS *fn* $e^{\wedge fn} = f(e^{\wedge Vin})$ where *Ic* terms appear only on the LHS, and *Vin* terms appear only on the RHS. We can now apply the Lambert W inverse to both sides giving

$$\left(Ic+Is\right)\frac{Req}{Vt} = W\left[\frac{Is_{Req}}{Vt} \cdot e^{\left(\left(\frac{Rf+Rc}{R_{TOT}\, Vt}\right)Vin + \left(\frac{Rin}{R_{TOT}\, Vt}\right)Vcc + \frac{Is\, Req}{Vt}\right)}\right] \qquad (39)$$

then solving for Ic gives

$$Ic = \frac{Vt}{Req} W\left[\frac{Is\, Req}{Vt} e^{\left(\left(\frac{Rf+Rc}{R_{TOT}\, Vt}\right)Vin + \left(\frac{Rin}{R_{TOT}\, Vt}\right)Vcc + \frac{Is\, Req}{Vt}\right)}\right] - Is \quad => \qquad (40)$$

Simplify using Vk

$$=> Ic = \frac{Vt}{Req}\cdot W\left[e^{\left(\frac{\left(\frac{Rf+Rc}{R_{Tot}}\right)Vin - Vk + \frac{Rin}{R_{Tot}}Vcc}{Vt}\right)}\right] - Is \quad (41)$$

Summary: the solution for Ic is

$$Ic = \frac{Vt}{Req}\cdot W\left[e^{\left(\frac{\left(\frac{Rf+Rc}{R_{Tot}}\right)Vin - Vk + \frac{Rin}{R_{Tot}}Vcc}{Vt}\right)}\right] - Is \quad \text{where} \quad (42)$$

$$Vk = Vt\cdot\ln\left(\frac{Vt}{Is\cdot Req}\right) - Is\ Req \qquad Req = \frac{Rin}{R_{TOT}}\left(Rc\left(\frac{Beta+1}{Beta}\right) + \frac{Rf}{Beta}\right)$$

LTspice plot **Figure 104b** show this solution is correct. Plot key: Rc stepped 300, 1k, 3k. LTspice is IcQ1, V(vc) while equations are I(B5), V(vp1). Vcc=10V, Rin=1k, Rf=10k. Beta=100.

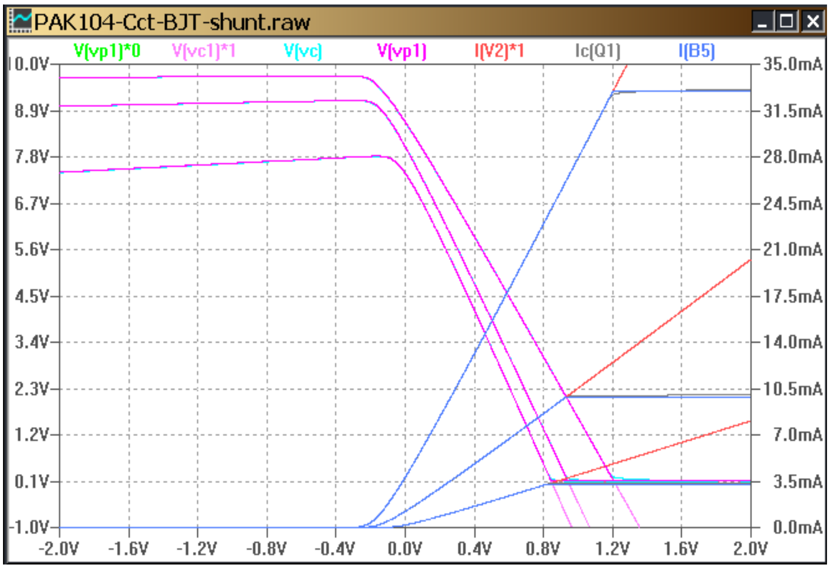

Figure 104b. Banwell's CE with shunt feedback and Rc stepped with the LTspice BJT

After a tedious process of analysis the solution is relatively simple. The seemingly complicated interactions in this circuit actually reduce to a few key combinations of resistors:

1) Req,
2) (Rf+Rc)/R_{TOT},
2) Rin/R_{TOT}
3) Rf/Rin

The ratio **Rf/Rin** is the key gain term and is discussed below

It is possible to use small signal analysis to arrive at a similar result for most of the range of currents used in amplifiers. But to do small-signal analysis you still need to guess an operating point, normally by using a guestimate for Vbe ~ 0.7V to find the Ic operating point then use r_e = 26mV/Ic. To establish the transistors approximate g_{fs} or gm.

The large-signal Lambert W solution, although tedious, has the advantage of providing the DC operating point as a general solution. Also all the circuit voltage and currents can be known from the Ic solution. Also, gain (gm and Av) can be estimated from this Ic solution by differentiation (see **PAK117**). Also Harmonic distortion (2^{nd} & 3^{rd} harmonics) can be estimated.

Calculating the output voltage

Having solved one current Ic we can now find any other current, voltage, gain, power or whatever. Vout is now found by back substitution into the original equations.

For the output voltage we use

$$Vout = Vcc - Iin \cdot Rc - Ic \cdot Rc - \frac{Ic}{Beta} \cdot Rc \quad \text{where} \quad Iin = \frac{Vbe - Vin}{Rin} \quad \& \quad Vbe = Vt \cdot \ln\left(\frac{Ic}{Is} + 1\right)$$

which gives

$$Vout = Vcc - \frac{Rc}{Rin} Vt \cdot \ln\left(\frac{Ic}{Is} + 1\right) - \frac{Rc}{Rin} Vin - Rc\left(\frac{Beta+1}{Beta}\right) Ic$$

Summary:

$$Vout = Vcc - \left(\frac{Beta+1}{Beta}\right) Rc \cdot Ic \; - \; \frac{Rc}{Rin}\left(Vin + Vt \cdot \ln\left(\frac{Ic}{Is} + 1\right)\right) \qquad 53(43)$$

Adding Vce saturation using the Max and Min functions

Figure 105b shows a plot for Ic and Vout with Re stepped that includes Ic and Vce saturation.

Equation (58) was modified to allow a large reverse input voltage. The Ln(...) term runs out of numeric range when Vin < -1.2V in LTspice. The following equation fixes this:

$$Vbe \simeq Min\left(Max\left(Vt \cdot \ln\left(\frac{Ic}{Is} + 1\right), 0\right), \; \frac{Rf + Rc}{R_{TOT}} (Vin - Vx) \; + (Vk + Is \cdot Req)\right) \qquad (44)$$

$$\text{where} \quad Vx = \frac{\left(Vk - \left(Rin / R_{TOT}\right) Vcc\right)}{(Rf + Rc) / R_{TOT}}$$

The *Vx* term is the input voltage where the Ic's W(e^{x}) part is W(e^0) and solving for Vin. The inner *Max*(*x,y*) function stops using the *Ln* (*fn* (*Ic*)) when Vbe passes through zero and the *Max*(*x,*0) function keeps it at zero rather than switching to an unwanted large negative limit when overflow occurs. The other function (a straight line) then takes control using the outer *Min* function. The plot in Figure 4.2a is extended to -2V to show that Vc is now correctly calculated beyond -1.2V. Saturation for Ic and Vc was obtained using

$$Ic = Min(Ic, IcSat) \quad \& \quad Ic_{Sat} = \frac{(Vcc - Vt \cdot \ln(Beta))}{Rc}$$

$$Vc = Max(Vc, Vt \cdot \ln(Beta))$$

Next, PAK105 demonstrates adding an **emitter resistor Re** to the previous Banwell circuit.

PAK Project – Course Material

PAK105 – Banwell's CE solution with shunt feedback & emitter degeneration

 8 Jan 2019 ver 0v5

CE with Shunt Voltage Feedback & emitter degeneration

The circuit **Figure 105a** below now includes *Re* . Changes to the equations are highlighted.

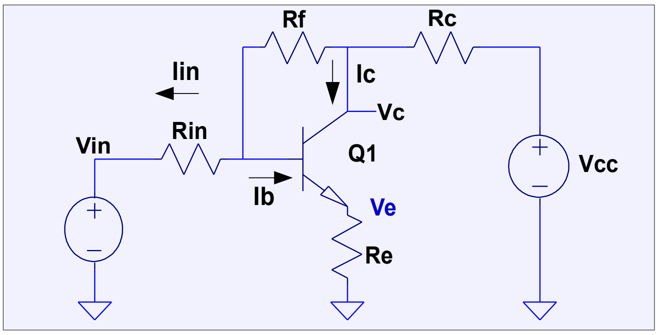

Figure 105a. Common Emitter amplifier with base to collector and emitter resistors.

Three equations are written. First the input voltage to the base node: For *Rin* write

$$(Vbe+Ve)-Vin=Iin\cdot Rin \quad \text{where} \quad Ve=Ie\cdot Re \qquad 37(45)$$

Note *Iin* is defined as flowing from the power source through *Rc* and *Rf*. Although voltage loops can be defined arbitrarily, this an intuitive approach (based on the power source). Repeating this for the voltage across *Rf* due to current *Iin* plus *Ib :*

$$\text{for } Rf \ldots \quad V_{BC}=Vc-(Vbe+Ve)=Iin\cdot Rf+\frac{Ic}{Beta}\cdot Rf \qquad 38(46)$$

$$\text{for } Rc \ldots \quad V_{Rc}=Vcc-Vc=Iin\cdot Rc+Ic\cdot Rc+\frac{Ic}{Beta}\cdot Rc \qquad 39(47)$$

We first eliminate *Vc* using (38) and (39) then eliminate *Iin* using (37). Rearrange (38) and (39)

$$Vc=(Vbe+Ve)-Iin\cdot Rf-\frac{Ic}{Beta}\cdot Rf \quad \text{and} \quad Vc=Vcc-Iin\cdot Rc-Ic\cdot Rc-\frac{Ic}{Beta}\cdot Rc$$

$$\text{equating} \quad (Vbe+Ve)-Iin\cdot Rf-\frac{Ic}{Beta}\cdot Rf=Vcc-Iin\cdot Rc-Ic\cdot Rc-\frac{Ic}{Beta}\cdot Rc$$

$$\text{collecting terms} \ldots \quad Ic\left(Rc+\frac{Rf+Rc}{Beta}\right)=Vcc-(Vbe+Ve)-Iin(Rf+Rc)$$

Substitute $Rc'=Rc+(Rf+Rc)/Beta$. Now eliminate *Iin*

$$Ic\,Rc'=Vcc-(Vbe+Ve)-\left(\frac{Rf+Rc}{Rin}\right)((Vbe+Ve)-Vin)$$

Since *Ve* is *Ie·Re=Ic·Re(Beta+1)/Beta* we can now collect *Ve* into *Ic*. First get *Ve* to LHS

$$Ic\,Rc'+Ve\left(1+\frac{Rf+Rc}{Rin}\right)=Vcc-\left(1+\frac{Rf+Rc}{Rin}\right)Vbe-\left(\frac{Rf+Rc}{Rin}\right)Vin$$

then substitute *Ve* and substitute $R_{TOT}=Rin+Rf+Rc$

$$Ic\,Rc'+Ic\,Re\left(\frac{Beta+1}{Beta}\right)\left(\frac{R_{TOT}}{Rin}\right)=Vcc-\left(\frac{R_{Tot}}{Rin}\right)Vbe-\left(\frac{Rf+Rc}{Rin}\right)Vin \quad =>$$

$$Ic\left(Rc'+Re\frac{R_{TOT}}{Rin}\left(\frac{Beta+1}{Beta}\right)\right)=Vcc-\left(\frac{R_{Tot}}{Rin}\right)Vbe-\left(\frac{R_{TOT}}{Rin}\right)Vin \quad \text{simplify}$$

$$Ic\,Rc'' = Vcc - \left(\frac{R_{TOT}}{Rin}\right)Vbe - \left(\frac{R_{TOT}}{Rin}\right)Vin \quad \text{where}$$
$$Rc'' = Rc' + Re\frac{R_{TOT}}{Rin}\left(\frac{Beta+1}{Beta}\right) \qquad 40(48)$$

Next, *Vbe* can be eliminated using the log form of the Shockley diode Equation, giving

$$Ic\,Rc'' = Vcc - \left(\frac{R_{TOT}}{Rin}\right)Vt\cdot\ln\left(\frac{Ic}{Is}+1\right) - \left(\frac{Rf+Rc}{Rin}\right)Vin \qquad 41(49)$$

We now proceed to get the above equation into the form of $fn\ e^{\wedge fn} = fn(e^{\wedge Vin})$ where *Ic* terms appear on the LHS, and *Vin* terms appear on the RHS.

Moving terms in front of the log gives

$$\ln\left(\frac{Ic}{Is}+1\right) + Ic\,Rc''\left(\frac{Rin}{R_{TOT}\,Vt}\right) = -\left(\frac{Rin}{R_{TOT}\ Vt}\right)\left(\left(\frac{Rf+Rc}{Rin}\right)Vin + Vcc\right) \quad => \qquad (50)$$

$$\ln\left(\frac{Ic+Is}{Is}\right) + Ic\frac{Rc''\,Rin}{R_{TOT}\,n\cdot Vt} = -\left(\left(\frac{Rf+Rc}{R_{TOT}\ Vt}\right)Vin + \left(\frac{Rin}{R_{TOT}\ Vt}\right)Vcc\right) \qquad (51)$$

Convert to exponential form (exponentiate both sides)

$$\left(\frac{Ic+Is}{Is}\right)\cdot\exp\left[Ic\frac{Rc''\,Rin}{R_{TOT}\,Vt}\right] = \exp\left[-\left(\left(\frac{Rf+Rc}{R_{TOT}\,Vt}\right)Vin + \left(\frac{Rin}{R_{TOT}\,Vt}\right)Vcc\right)\right] \quad =>$$
$$(Ic+Is)\cdot\exp\left[Ic\frac{Req}{Vt}\right] = Is\cdot\exp\left[-\left(\left(\frac{Rf+Rc}{R_{TOT}\,Vt}\right)Vin + \left(\frac{Rin}{R_{TOT}\,Vt}\right)Vcc\right)\right] \qquad (52)$$

Add an *Is* to the exponent (multiply both sides by exp[Is·Req / Vt])

$$(Ic+Is)\cdot e^{\left((Ic+Is)\frac{Req}{Vt}\right)} = Is\cdot e^{-\left(\left(\frac{Rf+Rc}{R_{TOT}Vt}\right)Vin + \left(\frac{Rin}{R_{TOT}Vt}\right)Vcc + \frac{Is\,Req}{Vt}\right)} \quad \text{where}$$
$$Req = \frac{Rin}{R_{Tot}}\left(Rc\left(\frac{Beta+1}{Beta}\right) + \frac{Rf}{Beta}\right) + Re\left(\frac{Beta+1}{Beta}\right) \qquad (53)$$

Balance the mantissa by multiplying both sides by *Req / Vt*

$$(Ic+Is)\frac{Req}{n\cdot Vt}\cdot e^{\left((Ic+Is)\frac{Req}{Vt}\right)} = \frac{Is\,Req}{Vt}\cdot e^{-\left(\left(\frac{Rf+Rc}{R_{TOT}Vt}\right)Vin + \left(\frac{Rin}{R_{TOT}Vt}\right)Vcc + \frac{Is\,Req}{Vt}\right)} \qquad (54)$$

We have balanced the LHS $fn\ e^{\wedge fn} = f(e^{\wedge Vin})$ where *Ic* terms appear only on the LHS, and *Vin* terms appear only on the RHS. We can now apply the Lambert W inverse to both sides giving

$$(Ic+Is)\frac{Req}{Vt} = W\left[\frac{Is_{Req}}{Vt}\cdot e^{\left(\left(\frac{Rf+Rc}{R_{TOT}Vt}\right)Vin + \left(\frac{Rin}{R_{TOT}Vt}\right)Vcc + \frac{Is\,Req}{Vt}\right)}\right] \qquad (55)$$

then solving for Ic gives

$$Ic = \frac{Vt}{Req}W\left[\frac{Is\,Req}{Vt}e^{\left(\left(\frac{Rf+Rc}{R_{TOT}Vt}\right)Vin + \left(\frac{Rin}{R_{TOT}Vt}\right)Vcc + \frac{Is\,Req}{Vt}\right)}\right] - Is \quad => \qquad (56)$$

Simplify using Vk

$$=> Ic = \frac{Vt}{Req} \cdot W\left[e^{\left(\frac{\left(\frac{Rf+Rc}{R_{Tot}}\right)Vin - Vk + \frac{Rin}{R_{Tot}}Vcc}{Vt}\right)}\right] - Is \quad (57)$$

Summary: the solution for Ic now with Re is ...

$$Ic = \frac{Vt}{Req} \cdot W\left[\exp\left(\frac{\left(\frac{Rf+Rc}{R_{Tot}}\right)Vin - Vk + \frac{Rin}{R_{Tot}}Vcc}{Vt}\right)\right] - Is \quad \text{where} \quad (58)$$

$$Vk = Vt \cdot \ln\left(\frac{Vt}{Is \cdot Req}\right) - Is\, Req \qquad Req = Re\left(\frac{Beta+1}{Beta}\right) + \frac{Rin}{R_{TOT}}\left(Rc\left(\frac{Beta+1}{Beta}\right) + \frac{Rf}{Beta}\right)$$

LTspice plot **Figure 105b** show this solution is correct. Plot key: Rf & Rin with Re stepped 1mR, 10R and 100R. LTspice is red & brown, while equations are gold & cyan. Vcc=10V, Rc=Rin=1k, Rf=10k. Beta=100.

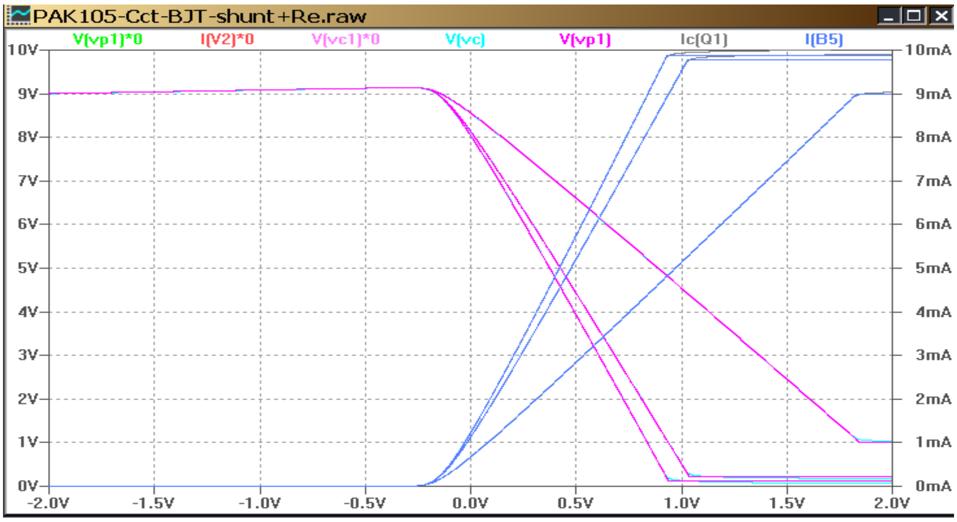

Figure 105b. Banwell's CE with shunt feedback & Re with the LTspice BJT

After a tedious process of analysis the solution is relatively simple. The seemingly complicated interactions in this circuit actually reduce to a few key combinations of resistors:

1) Req,
2) (Rf+Rc)/R_{TOT},
2) Rin/R_{TOT}
3) Rf/Rin

The ratio **Rf/Rin** is the key gain term and is discussed below. Adding Re changes Req.

It is possible to use small signal analysis to arrive at a similar result for most of the range of currents used in amplifiers. But to do small-signal analysis you still need to guess an operating point, normally by using a guestimate for Vbe ~ 0.7V to find the Ic operating point then use r_e = 26mV/Ic. To establish the transistors approximate g_{fs} or gm.

The large-signal Lambert W solution, although tedious, has the advantage of providing the DC operating point as a general solution. Also all the circuit voltage and currents can be known from the Ic solution. Also, gain (gm and Av) can be estimated from this Ic solution by differentiation (see **PAK117**). Also Harmonic distortion (2nd & 3rd harmonics) can be estimated.

Calculating the output voltage

Having solved one current Ic we can now find any other current, voltage, gain, power or whatever. Vout is now found by back substitution into the original equations. For the output voltage we can use

$$Vout = Vcc - Iin\cdot Rc - Ic\cdot Rc - \frac{Ic}{Beta}\cdot Rc \quad \text{where} \quad Iin = \frac{Vbe + Ve - Vin}{Rin} \quad \text{and}$$

$$Vbe = Vt\cdot\ln\left(\frac{Ic}{Is}+1\right) \quad \text{and} \quad Ve = Ie\cdot Re = Ic\cdot Re\left(\frac{Beta+1}{Beta}\right)$$

which gives

$$Vout = Vcc - \frac{Rc}{Rin}Vt\cdot\ln\left(\frac{Ic}{Is}+1\right) - \frac{Rc\cdot Re}{Rin}\left(\frac{Beta+1}{Beta}\right)Ic - \frac{Rc}{Rin}Vin - Rc\left(\frac{Beta+1}{Beta}\right)Ic$$

$$Vout = Vcc - \left(1+\frac{Re}{Rin}\right)\left(\frac{Beta+1}{Beta}\right)Rc\cdot Ic \; - \; \frac{Rc}{Rin}\left(Vin + Vt\cdot\ln\left(\frac{Ic}{Is}+1\right)\right) \qquad (59)$$

Adding Vce saturation using the Max and Min functions

Figure 105b shows a plot for Ic and Vout with Re stepped that includes Ic and Vce saturation.

Equation (58) was modified to allow a large reverse input voltage. The Ln(...) term runs out of numeric range when Vin < -1.2V in LTspice. The following equation fixes this:

$$Vbe \simeq Min\left(Max\left(n\cdot Vt\cdot\ln\left(\frac{Ic}{Is}+1\right),0\right),\; \frac{Rf+Rc}{R_{TOT}}(Vin - Vx) \; + (Vk + Is\cdot Req)\right)$$

$$\text{where} \quad Vx = \frac{\left(Vk - \left(Rin/R_{TOT}\right)Vcc\right)}{(Rf+Rc)/R_{TOT}} \qquad (60)$$

The *Vx* term is the input voltage where the Ic's W(e^{x}) part is W(e^0) and solving for Vin. The inner *Max*(*x,y*) function stops using the *Ln* (*fn* (*Ic*)) when Vbe passes through zero and the *Max*(*x,*0) function keeps it at zero rather than switching to an unwanted large negative limit when overflow occurs. The other function (a straight line) then takes control using the outer *Min* function. The plot in Figure 4.2a is extended to -2V to show that Vc is now correctly calculated beyond -1.2V. Saturation for Ic and Vc was obtained using

$$Ic = Min(Ic, IcSat) \quad \& \quad Ic_{Sat} = \frac{(Vcc - Vt\cdot\ln(Beta))}{Re + Rc}$$

$$Vc = Max(Vc, IcSat\cdot Re + Vt\cdot\ln(Beta))$$

PAK Project – Course Material

PAK106 – General solution for y·e^y=e^x and y+Ln(y)=x

 8 Jan 2019 ver 0v5

General solution for y·e^y=e^x and y+Ln(y)=x

Diode and transistors with resistors have the form
y+Ln(y)=x
from the series voltage loop equations $I_D \cdot R + Vt \cdot Ln(I_D) = Vin$.
Refer to PAK102 section 'D+R equation solving steps'.

For algebraic manipulation I find it better to converted the Log form to the exponential form:
$y\cdot e^y = e^x$. The log form is converted by exponentiating both sides.

The general solution for $(y+b)e^{k\cdot y} = e^x$ or $\ln(y+b)+k\cdot y = x$

Normalise the y term in the mantissa

$$(y+b)e^{k\cdot y} = e^x \quad => \quad (y+b)e^{k\cdot y} = e^x \quad => \quad (y+b)e^{k\cdot y} = e^x \qquad (61)$$

Multiply both sides by exp(b/k), then both sides by k

$$(y+b)e^{\frac{k}{1}y}\cdot e^{\left(\frac{b}{k}\right)} = \left(\frac{1}{a}\right)e^x\cdot e^{(b)} \quad => \quad k(y+b)\cdot e^{k(y+b)} = k\cdot e^{(x+b)} \qquad (62)$$

We can now write the inverse

$$k(y+b) = W\left[k\cdot e^{(x+b)}\right] \quad => \quad k(y+b) = W\left[e^{(x+b+\ln(k))}\right] \qquad (63)$$

Summary

$$(y+b)e^{k\cdot y} = e^x \quad => \quad y=\frac{1}{k}\cdot W\left[e^{x+b+\ln()}\right]-b \qquad (64)$$

Fun exercise: Using a feedback loop to solve for Wx

Figure 106a shows an LTspice circuit to solve a D+R equation for Wx using a feedback loop.

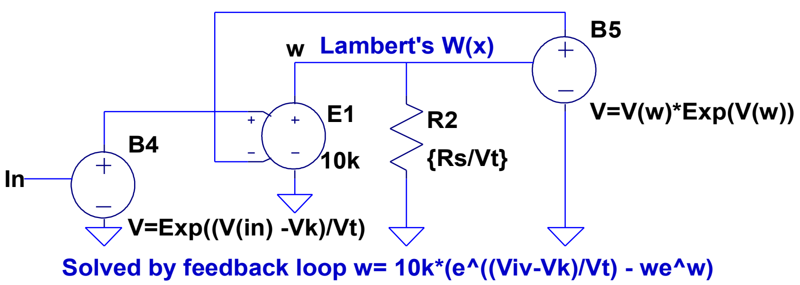

Fig 106a. LTspice circuit to solve a D+R equation for Wx using a feedback loop

This feedback loop tries to find V(w) so $V(w)\cdot e^{V(w)} = e^{\left(\frac{(V(in)-Vk)}{Vt}\right)}$. It works for stepped DC runs. But a .trans run steps very slowly from about 0.6V where the LHS is about 40 volts. Run the file PAK106-Cct-we^w-loop. Can you can make it run faster?

PAK Project – Course Material

PAK107 – General solution for y·e^y^2=e^x and y^2+Ln(y)=x

 8 Jan 2019 ver 0v5

General solution for y·e^y^2=e^x and y^2+Ln(y)=x

[Ref Ortiz-Conde EDS Feb 2006]

The solution to $y\cdot e^{k\,y^2} = e^x$ or $\ln(y)+k\cdot y^2 = x$ (65)

Using the exponential form, first square both sides, then multiply by 2,

=> $y^2\cdot e^{2k\,y^2} = e^{2x}$ => $2k\,y^2\cdot e^{2k\,y^2} = 2k\cdot e^{2x}$ $2k\cdot y^2 = W\left[2k\cdot e^{2x}\right]$ (66)

this gives an inverse

=> $2k\cdot y^2 = W\left[2k\cdot e^{2x}\right]$ => $y = \sqrt{\frac{1}{2k}}\sqrt{W\left[2k\cdot e^{2x}\right]}$ (67)

Summary

$y\cdot e^{k\,y^2} = e^x$ => $y = \sqrt{\frac{1}{2k}}\sqrt{W\left[2k\cdot e^{2x}\right]}$ or $y = \sqrt{\frac{1}{2k}}\sqrt{W\left[e^{2x+\ln(2k)}\right]}$ (68)

Corollary:

Solving $y\cdot e^{k\,y^m} = e^x$ or $\ln(y)+k\cdot y^m = x$ simply replace all 2's with 'm' gives:

$y\cdot e^{k\,y^m} = e^x$ => $y = \sqrt{\frac{1}{m\cdot k}}\sqrt{W\left[m\cdot k\cdot e^{m\cdot x}\right]}$ or $y = \sqrt{\frac{1}{m\cdot k}}\sqrt{W\left[e^{m\cdot x+\ln(m\cdot k)}\right]}$ (69)

This can be used to solve a nonlinear resistor with a diode $Vt\cdot\ln\left(\frac{I}{Is}\right)+Ro\cdot\left(\frac{I}{Io}\right)^m = Vin$

PAK Project – Course Material

PAK108 – General solution for y^2·e^y=e^x and y +2·Ln(y)=x

 8 Jan 2019 ver 0v5

General solution for y^2·e^y=e^x and y +2·Ln(y)=x

[Ref Ortiz-Conde EDS Feb 2006]

The solution to $y^2 \cdot e^{k\,y} = e^x$ or $2 \cdot \ln(y) + k \cdot y = x$ (70)

Using the exponential form, square-root both sides, then multiply by 1/2, gives an inverse

$$=> \quad y \cdot e^{\frac{k\,y}{2}} = e^{\frac{x}{2}} \quad => \quad \frac{k\,y}{2} \cdot e^{\frac{k\,y}{2}} = \frac{k}{2} \cdot e^{\frac{x}{2}} \quad => \qquad (71)$$

Take inverse. Solve for y

$$=> \quad \frac{k\,y}{2} = W\left[\frac{k}{2} \cdot e^{\frac{x}{2}}\right] \quad => \quad y = \frac{2}{k} \cdot W\left[\frac{k}{2} \cdot e^{\frac{x}{2}}\right] \qquad (72)$$

Summary

$$y^2 \cdot e^{k\,y} = e^x \quad => \quad y = \frac{2}{k} \cdot W\left[\frac{k}{2} \cdot e^{\frac{x}{2}}\right] \quad \text{or} \quad y = \frac{2}{k} W\left[e^{\frac{x}{2} + \ln\left(\frac{k}{2}\right)}\right] \qquad (73)$$

Corollary:

Solving $y^2 \cdot e^{k\,y} = e^x$ or $m \cdot \ln(y) + k \cdot y = x$ or $\ln(y^m) + k \cdot y = x$

simply replace all 2's with 'm' gives:

$$y^m \cdot e^{k\,y} = e^x \quad => \quad y = \frac{m}{k} \cdot W\left[\frac{k}{m} \cdot e^{\frac{x}{m}}\right] \quad \text{or} \quad y = \frac{m}{k} W\left[e^{\frac{x}{m} + \ln\left(\frac{k}{m}\right)}\right] \qquad (74)$$

PAK Project – Course Material

PAK109 – General solution for (1/y)e^y=e^x and y–Ln(y)=x

 8 Jan 2019 ver 0v5

General solution for (1/y)e^y=e^x and y–Ln(y)=x

[Ref Ortiz-Conde EDS Feb 2006]

The solution to $(1/y)e^{k\,y}=e^{x}$ or $k\cdot y-\ln(y)=x$ (75)

Inverse all

$$ye^{-k\,y}=e^{-x} \quad => \quad -ky\cdot e^{-k\,y}=-k\cdot e^{-x} \tag{76}$$

Apply inverse

$$-ky=W\left[-k\cdot e^{-x}\right] \quad => \quad y=-k\cdot W\left[-k\cdot e^{-x}\right] \tag{77}$$

Simplify

$$=> \quad y=-k\cdot W\left[-e^{-x+\ln(k)}\right] \tag{78}$$

Summary

$$(1/y)e^{k\,y}=e^{x} \quad => \quad y=-k\cdot W\left[-e^{-x+\ln(k)}\right] \tag{79}$$

The negative branch for W(-x) only has real values for x≥-1/e.
Also for x negative in the range for 0>x≥-1/e there are two real solutions. This region can be used for bi-stable circuits that use positive feedback. The Lambert W-function can be used for analytical stability feedback systems [ref Ulsoy 2009].

PAK Project – Course Material

PAK110 – General solution for (1/y^2)e^y=e^x and y–Ln(2y)=x

 8 Jan 2019 ver 0v5

PAK Project

General solution for (1/y^2)e^y=e^x and y–Ln(2y)=x

[Ref Ortiz-Conde EDS Feb 2006]

The solution to $\frac{1}{y^2}\cdot e^{k\,y} = e^x$ or $k\cdot y - \ln(2y) = x$ (80)

Take square-root

$$\frac{1}{y}\cdot e^{\frac{k\,y}{2}} = e^{\frac{x}{2}} \quad => \quad \frac{1}{y}\cdot e^{\frac{k\,y}{2}} = \frac{1}{2}\cdot e^{\frac{x}{2}} \tag{81}$$

Inverse, then divide by -2 and times k

$$y\cdot e^{\frac{-k\,y}{2}} = e^{\frac{-x}{2}} \quad => \quad \frac{-k\,y}{2}\cdot e^{\frac{-k\,y}{2}} = \frac{-k}{2}\cdot e^{\frac{-x}{2}} \tag{82}$$

Apply inverse

$$\frac{-k\,y}{2} = W\left[\frac{-k}{2}\cdot e^{\frac{-x}{2}}\right] \quad => \quad y = \frac{-2}{k}\cdot W\left[\frac{-k}{2}\cdot e^{\frac{-x}{2}}\right] \tag{83}$$

Simplify

$$=> \quad y = \frac{-2}{k}\cdot W\left[-e^{\frac{-x}{2}+\ln(k)-\ln(2)}\right] \tag{84}$$

Summary

$$\frac{1}{y^2}\cdot e^{k\,y} = e^x \quad => \quad y = \frac{-2}{k}\cdot W\left[-e^{\frac{-x}{2}+\ln(k)-\ln(2)}\right] \tag{85}$$

The negative branch for W(-x) only has real values for x≥-1/e.
Also for x negative in the range for 0>x≥-1/e there are two real solutions. This region can be used for bi-stable circuits that use positive feedback. The Lambert W-function can be used for analytical stability feedback systems [ref Ulsoy 2009].

Corollary:

Solving $\frac{1}{y^m}\cdot e^{k\,y} = e^x$ or $k\cdot y - \ln(m\cdot y) = x$

simply replace all 2's with 'm' gives:

$$\frac{1}{y^m}\cdot e^{k\,y} = e^x \quad => \quad y = \frac{-m}{k}\cdot W\left[-e^{\frac{-x}{m}+\ln(k)-\ln(m)}\right] \tag{86}$$

PAK Project – Course Material

PAK111 – General solution for y+e^y=e^x and e^y·e^e^y=e^e^x and (x−y)+Ln(x−y)=x

 8 Jan 2019 ver 0v5

PAK Project

General solution for y+e^y=e^x and e^y·e^e^y= e^e^x and (x−y)+Ln(x−y)=x

[Ref Ortiz-Conde EDS Feb 2006]

First procedure

The solution to

$$y+e^{k\,y}=e^{x} \quad \text{Let e^x=X} \Rightarrow \quad y+e^{k\,y}=X \quad \Rightarrow \quad k\cdot y+k\cdot e^{k\,y}=k\cdot X \tag{87}$$

Rearrange

$$k(X-y)=e^{k\,y} \quad \Rightarrow \quad k(X-y)=\frac{e^{k\,y}}{e^{k\,X}}e^{k\,X} \quad \Rightarrow \quad k(X-y)=e^{k(y-X)}e^{k\,X} \tag{88}$$

Take inverse then solve for y

$$k(X-y)e^{k(X-y)}=e^{k\,X} \quad \Rightarrow \quad k(X-y)=W\left[e^{k\,X}\right] \quad \Rightarrow \quad X-y=\frac{1}{k}\cdot W\left[e^{k\,X}\right] \tag{89}$$

Summary

$$y+e^{k\,y}=e^{x} \quad \Rightarrow \quad y=e^{x}-\frac{1}{k}\cdot W\left[e^{k\,e^{x}}\right] \tag{90}$$

Second procedure

The solution to

$$y+e^{k\,y}=e^{x} \quad \Rightarrow \quad e^{y}\cdot e^{\left(e^{k\,y}\right)}=e^{\left(e^{x}\right)} \quad \text{Let e^x=X} \tag{91}$$

Multiply both sides by e^k , then both sides by k

$$e^{k\,y}\cdot e^{\left(k\,e^{k\,y}\right)}=e^{(k\cdot X)} \qquad k\cdot e^{k\,y}\cdot e^{\left(k\,e^{k\,y}\right)}=k\cdot e^{(k\cdot X)} \quad \Rightarrow \tag{92}$$

Take inverse then solve for y

$$k\cdot e^{k\,y}=W\left[k\cdot e^{(k\cdot X)}\right] \quad \Rightarrow \quad y=\frac{1}{k}\ln\left(\frac{1}{k}\cdot W\left[k\cdot e^{(k\cdot X)}\right]\right) \tag{93}$$

Summary

$$y+e^{k\,y}=e^{x} \quad \Rightarrow \quad y=\frac{1}{k}\ln\left(\frac{1}{k}\cdot W\left[k\cdot e^{\left(k\cdot e^{x}\right)}\right]\right) \tag{94}$$

We have two answers. Both must be equivalent so we can choose either

$$y+e^{k\,y}=e^{x} \quad \Rightarrow \quad y=e^{x}-\frac{1}{k}\cdot W\left[e^{k\,e^{x}}\right]=\frac{1}{k}\ln\left(\frac{1}{k}\cdot W\left[k\cdot e^{\left(k\cdot e^{x}\right)}\right]\right) \tag{95}$$

The simpler one that does not require Ln(...) gives better numerical ranging.

Corollary:

The two solutions leads us to an equivalence relationship

$$\ln\left(W\left[e^{X}\right]\right) \equiv X-W\left[e^{X}\right] \quad \text{or} \quad W\left[e^{X}\right]=X-\ln\left(W\left[e^{X}\right]\right) \tag{96}$$

PAK Project – Course Material

PAK112 – General solution for (1/y)Ln(y)=x

General solution for (1/y)Ln(y)=x or Ln(Ln(y))–Ln(y)=ln(x)

[Ref Ortiz-Conde EDS Feb 2006]

The solution to $\left(\frac{1}{y}\right)\ln(y) = x$ or $\ln\left(\sqrt[y]{y}\right) = x$ or $\ln(\ln(y)) - \ln(y) = \ln(x)$ (97)

Move y to RHS. Then exponentiate

$$\left(\frac{1}{y}\right)\ln(y) = x \quad => \quad \ln(y) = x \cdot y \quad => \quad y = e^{x \cdot y} \tag{98}$$

Inverse. Then times -X y

$$\frac{1}{y} = e^{-x \cdot y} \quad => \quad -x = (-x \cdot y) \cdot e^{-x \cdot y} \tag{99}$$

Apply inverse

$$-x \cdot y = W[-x] \quad => \quad y = -\frac{1}{x} W[-x] \tag{100}$$

Summary

$$\left(\frac{1}{y}\right)\ln(y) = x \quad => \quad y = -\frac{1}{x} \cdot W[-x] \tag{101}$$

PAK Project – Course Material

PAK113 – Solution (ay^2+by+c)e^y(1+dy)=e^x or y(1+dy)+Ln(ay^2+by+c)=x

 8 Jan 2019 ver 0v5

Solution for (ay^2+by+c)e^ky(1+dy)=e^x & ky(1+dy)+Ln(ay^2+by+c)=x

For $\left(a\,y^2+b\,y+c\right)e^{k\cdot y(1+d\,y)}=e^x$

Normalise b·y term assuming the y^2 part is a second order term:

$$\left(\frac{a}{b}y^2+y+\frac{c}{b}\right)e^{k\cdot y(1+d\,y)}=\frac{1}{b}e^x$$

$$\left(y\left(1+\frac{a}{b}y\right)+\frac{c}{b}\right)e^{k\cdot y(1+d\,y)}=\frac{1}{b}e^x \quad =>$$

$$\left(k\,y\left(1+\frac{a}{b}y\right)+\frac{k\,c}{b}\right)e^{k\cdot y(1+d\,y)}=\frac{k}{b}e^x$$

It is not possible to solve this exactly for y since a/b≠d *in general*. To get a general solution we need a (1+d·y) term in the LHS mantissa in stead of (1+(a/b)y). So divide both sides by (1+(a/b)y) then multiply both sides by (1+d·y). For simpler notation write $Sw=\frac{1+d\cdot y}{1+(a/b)y}$.

This method places a 'Sw' term on the RHS so an explicit solution is not possible general using the ordinary W-function. So recursions has to be used where an estimate Ic_{est} for Ic is used on the RHS in the Sw function.

Let $Sw=\frac{1+d\cdot y}{1+(a/b)y}$. Multiply both sides by Sw

$$\left(k\,Sw\,y\left(1+\frac{a}{b}y\right)+\frac{k\,c\,Sw}{b}\right)e^{k\cdot y(1+d\,y)}=\frac{k\,Sw}{b}e^x \quad =>$$

$$\left(k\,y(1+d\,y)+\frac{k\,c\,Sw}{a}\right)e^{k\cdot y(1+d\,y)}=\frac{k\,Sw}{a}e^x$$

Now add 'k·c·Sw/a' to the LHS and RHS exponents

$$\left(k\,y(1+d\,y)+\frac{k\,c\,Sw}{a}\right)e^{k\cdot y(1+d\,y)}\,e^{\frac{k\,c\,Sw}{a}}=\frac{k\,Sw}{a}e^x\,e^{\frac{k\,c\,Sw}{a}} \quad =>$$

$$\left(k\,y(1+d\,y)+\frac{k\,c\,Sw}{a}\right)e^{k\cdot y(1+d\,y)+\frac{k\,c\,Sw}{a}}=\frac{k\,Sw}{a}e^{x+\frac{k\,c\,Sw}{a}}$$

Take W inverse

$$k\,y(1+d\,y)+\frac{k\,c\,Sw}{a}=W\left[\frac{k\,Sw}{a}e^{x+\frac{k\,c\,Sw}{a}}\right] \quad =>$$

$$y(1+d\,y)+\frac{c\,Sw}{a}=\frac{1}{k}W\left[e^{x+\frac{c\cdot Sw}{a}+\ln\left(\frac{k\,c\,Sw}{a}\right)}\right] \quad =>$$

$$d\,y^2+y=\frac{1}{k}W\left[e^{x+\frac{c\cdot Sw}{a}+\ln\left(\frac{k\,c\,Sw}{a}\right)}\right]-\frac{c\,Sw}{a} \quad \text{divide by d (assume d≠0)}$$

$$y^2+\frac{1}{d}y=\frac{1}{k\,d}W\left[e^{x+\frac{c\cdot Sw}{a}+\ln\left(\frac{k\,c\,Sw}{a}\right)}\right]-\frac{c\,Sw}{a\,d} \quad \text{complete the square}$$

$$\left(y+\frac{1}{2d}\right)^2 = \frac{1}{k\,d}\,W\left[e^{x+\frac{c\cdot Sw}{a}+\ln\left(\frac{k\,c\,Sw}{a}\right)}\right]+\left(\frac{1}{2d}\right)^2-\frac{c\,Sw}{a\,d}$$

Square root and solve for y

$$y = -\frac{1}{2d}+\sqrt{\frac{1}{k\,d}W\left[e^{x+\frac{c\cdot Sw}{a}+\ln\left(\frac{k\,c\,Sw}{a}\right)}\right]+\left(\frac{1}{2d}\right)^2-\frac{c\,Sw}{a\,d}}$$ where $Sw=\frac{1+d\cdot y_{est}}{1+(a/b)\,y_{est}}$, d≠0.

The switch function 'Sw' starts at the value of Sw=1 and limits at Sw=b·d/a. It moves the knee voltage higher as y increases at the same time as the W-function moves from proportional to W to the square root of W for large y.

Solution with d=0 is solution to $\left(a\,y^2+b\,y+c\right)\,e^{k\cdot y} = e^{x}$

Back up a few steps and solve $y+\frac{c\,Sw}{a} = \frac{1}{k}\,W\left[e^{x+\frac{c\cdot Sw}{a}+\ln\left(\frac{k\,c\,Sw}{a}\right)}\right]$ =>

$$y = \frac{1}{k}W\left[e^{x+\frac{c\cdot Sw}{a}+\ln\left(\frac{k\,c\,Sw}{a}\right)}\right]-\frac{c\,Sw}{a}$$ where where $Sw=\frac{1}{1+(1/b)\,y_{est}}$

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PAK Project – Course Material

PAK114 – Derivatives dW(x)/dx and d^2W(x)/dx^2

 8 Jan 2019 ver 0v5

Derivatives dW(x)/dx and d^2W(x)/dx^2

The first derivative of W(e^x) [ref Corless 1996 p339]

$$\frac{d\,W(e^x)}{dx} = \frac{W(e^x)}{1+W(e^x)} \tag{102}$$

Derivation:

$$y\,e^y = e^x \quad <=> \quad y = W(e^x)$$

$$y\,e^y = e^x \quad => \quad y + \ln y = x \quad => \quad W(e^x) + \ln(W(e^x)) = x$$

$$\frac{d\,W(e^x)}{dx} + \frac{d\ln(W(e^x))}{dx} = \frac{dx}{dx} \quad => \quad \frac{d\,W(e^x)}{dx} + \frac{1}{W(e^x)}\frac{d\,W(e^x)}{dx} = 1 \quad =>$$

$$\frac{d\,W(e^x)}{dx}\left(1+\frac{1}{W(e^x)}\right) = 1 \quad => \quad \frac{d\,W(e^x)}{dx} = \frac{1}{1+\frac{1}{W(e^x)}} \quad =>$$

$$\frac{d\,W(e^x)}{dx} = \frac{W(e^x)}{1+W(e^x)}$$

Second derivative of W(e^x)

$$\frac{d^2 W(e^x)}{dx^2} = \frac{W(e^x)^2\left(2+W(e^x)\right)}{\left(1+W(e^x)\right)^3} \quad \text{or} \quad \frac{d^2 W(e^x)}{dx^2} = \frac{1}{\left(1+W(e^x)\right)^2}\cdot\frac{d\,W(e^x)}{dx} \tag{103}$$

Derivation

$$\frac{d^2 W(e^x)}{dx^2} = \frac{d\left\{\frac{W(e^x)}{1+W(e^x)}\right\}}{dx} \quad => \quad = \frac{\left(1+W(e^x)\right)\frac{d\,W(e^x)}{dx} - W(e^x)\frac{d\left(1+W(e^x)\right)}{dx}}{\left(1+W(e^x)\right)^2}$$

$$\frac{d^2 W(e^x)}{dx^2} = \frac{1}{\left(1+W(e^x)\right)^2}\cdot\frac{d\,W(e^x)}{dx} \quad => \quad \frac{d^2 W(e^x)}{dx^2} = \frac{1}{\left(1+W(e^x)\right)^2}\left\{\frac{W(e^x)}{1+W(e^x)}\right\}$$

$$\frac{d^2 W(e^x)}{dx^2} = \frac{W(e^x)}{\left(1+W(e^x)\right)^3}$$

Alternative form is

$$\frac{d^2 W(e^x)}{dx^2} = \frac{1}{\left(1+W(e^x)\right)^2}\cdot\frac{d\,W(e^x)}{dx} \tag{104}$$

PAK Project – Course Material

PAK115 – Indefinite integrals ∫W(x)dx and ∫W(e^x)dx

 8 Jan 2019 ver 0v5

Indefinite integrals ∫W(x)dx and ∫W(e^x)dx

1. Integral of W(*x*) [ref Corless 1996 p340]

$$\int W(x)\,dx = \left[W(x)^2 + W(x) + 1\right] e^{W(x)} + c$$

Derivation:

$\int W(x)dx = y$ let $p \equiv W(x)$ where *p* is shorthand for W(x) (change it back at the end).

So $x = W(x)\,e^{W(x)}$ becomes $x = p\,e^{p}$ also

$$\frac{dy}{dx} = \frac{d\int W(x)dx}{dx} = W(x) = p \quad \text{and} \quad \frac{dx}{dy} = \frac{1}{p}$$

Differentiate *x* with respect to *y* using the Chain Rule:

$$\frac{dx}{dy} = \frac{dx}{dp}\cdot\frac{dp}{dy} = \left(\frac{d\,p\cdot e^{p}}{dp}\right)\cdot\frac{dp}{dy} = \left(1\cdot e^{p} + p\,e^{p}\right)\cdot\frac{dp}{dy} = (p+1)\,e^{p}\cdot\frac{dp}{dy}$$

Substitute $\frac{dx}{dy} = \frac{1}{p}$ gives $\frac{1}{p} = (p+1)\,e^{p}\cdot\frac{dp}{dy}$ solve for $^{dy}/_{dp}$ gives $\frac{dy}{dp} = p(p+1)\,e^{p}$

Now integrate $y = \int p(p+1)\,e^{p}\,dp = \int p^2\,e^{p}\,dp + \int p\,e^{p}\,dp$ gives us two integrals.

The second integral is the easiest:

By parts $\int u\cdot v\,dv = u\cdot v - \int v\,du$ let u = *p* & v = e^*p* so du/d*p* = 1 & dv/d*p* = d(e^*p*)/d*p* = e^*p*

substituting parts $\int p\,e^{p}\,dp = p\cdot e^{p} - \int e^{p}\,dp = p\cdot e^{p} - e^{p} = (p-1)e^{p}$

Now the first integral $\int p^2\,e^{p}\,dp$ by parts: let *u* = *p*^2 and *v* = e^*p* so du/d*p* = 2*p* and dv/d*p* = d(e^*p*)/d*p* = e^*p* substituting parts $\int p^2\,e^{p}\,dp = p^2\cdot e^{p} - \int e^{p}\cdot 2\mathrm{p}\,dp$

where the remaining integral was done above, so

$$\int p^2\,e^{p}\,dp = p^2\cdot e^{p} - \int e^{p}\cdot 2\mathrm{p}\,dp = p^2\cdot e^{p} - 2(p-1)e^{p} = p^2\cdot e^{p} - (2\mathrm{p}-2)e^{p}$$

Now combining the two integrals gives

$$y = \int p^2\,e^{p}\,dp + \int p\,e^{p}\,dp = p^2\cdot e^{p} - (2\mathrm{p}-2)e^{p} + (p-1)e^{p} = (p^2 + p - 1)e^{p}$$

Substitute shorthand notation $W(x) = p$ plus constant of integration 'c' gives

$$y = \int W(x)\,dx + c = \left[W(x)^2 + W(x) - 1\right] e^{W(x)} + c$$

QED

2. Integral of W(e^x) [under construction]

Does $\int W[e^{x}]\,dx = \left(W(e^{x})^2 + W(e^{x}) + 1\right) e^{W(e^{x})} + c$?

...possibly since by the Cain Rule the derivative of $\frac{d(e^{x})}{dx} = e^{x}$ Maybe not for e^ax ???]

PAK Project – Course Material

PAK116 – Adding two W(e^x) terms into a single W(e^x) term

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Adding two W(e^x) terms into a single W(e^x) term

[Ref _]

$$W\left(e^{x1}\right)+W\left(e^{x2}\right)=W\left[\frac{e^{x1+x2}}{\left(\frac{1}{W\left(e^{x1}\right)}+\frac{1}{W\left(e^{x2}\right)}\right)^{-1}}\right]=W\left[\frac{e^{x_1+x_2}}{W\left(e^{x1}\right)//W\left(e^{x2}\right)}\right]=W\left[e^{x1+x2-\ln\left(W\left(e\wedge x2\right)//W\left(e\wedge x2\right)\right)}\right]$$

Derivation uses $e^{W(x)}=\frac{x}{W(x)}$ from $ye^{y}=x \iff y=W(x)$ so $W(x)e^{W(x)}=x$

Adding two scaled W(e^x) terms into a single W(e^x) term

$$a\cdot W\left(e^{x1}\right)+b\cdot W\left(e^{x2}\right)=W\left[\frac{a\,{x_1}^{a}\cdot b\,{x_2}^{b}}{\left(W\left(e^{x1}\right)\right)^{a-1}\left(W\left(e^{x2}\right)\right)^{b-1}}\left(a\cdot W\left(e^{x1}\right)//b\cdot W\left(e^{x2}\right)\right)\right]$$

PAK Project – Course Material

PAK117 – Calculating the differential voltage gain & HD

 8 Jan 2019 ver 0v5

4.2b Calculating the differential voltage gain [ref: Banwell 2000]

PAK104 and PAK105 covers the derivation of the *large signal* Ic for the CE with shunt feedback. Here we use differentiation to find the voltage gain of this circuit *for large signal.*

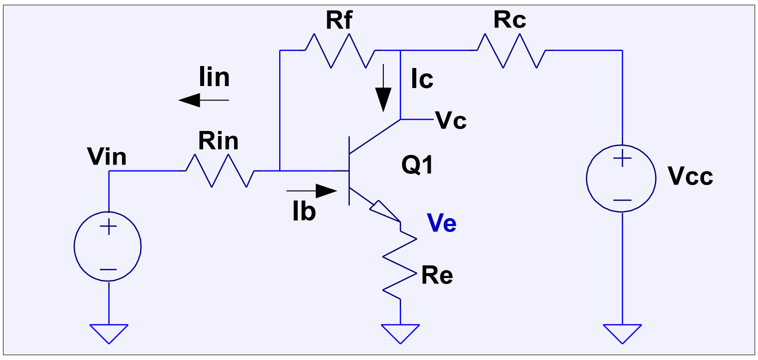

Fig.105a again. Banwell's CE with shunt feedback and Re

The large signal Ic solution for the CE with shunt feedback with Rs is:

$$Ic = \frac{Vt}{Req}\cdot W\left[\exp\left(\frac{\left(\frac{Rf+Rc}{R_{Tot}}\right)Vin - Vk + \frac{Rin}{R_{Tot}}Vcc}{Vt}\right)\right] - Is \quad \text{where} \qquad 101(105)$$

$$Vk = Vt\cdot\ln\left(\frac{Vt}{Is\cdot Req}\right) - Is\,Req \qquad Req = Re\left(\frac{Beta+1}{Beta}\right) + \frac{Rin}{R_{TOT}}\left(Rc\left(\frac{Beta+1}{Beta}\right) + \frac{Rf}{Beta}\right)$$

The differential voltage gain $\frac{dVc}{dVin}$ can be calculated for any Vin by differentiation of the Ic equation with respect to Vin, giving

$$Av_{CL} = \frac{d\,Vc}{d\,Vin} = -\left(\frac{Rf}{Rin} - \frac{1}{1+We(101)}\,\frac{Rin+Rf}{Rin}\,\frac{Rf+Rc}{R_{TOT}}\right) \qquad 102/(106)$$

where We(101) is the Lambert W terms in the Ic Equation (101). PAK114 gives the equation for the derivative of W(e^x).

Interestingly, the voltage gain is not defined explicitly by Re as we might expect since it alters the gm of the transistor and therefore the internal voltage gain, but it is affected by We(101)). Rf/Rin is the standard result for the gain limit of a transresistance stage and an inverting opamp.

4.2c Estimates of Distortion HD2 and HD3

The section above shows that we can now directly calculate the differential current gain and voltage gain of a complete BJT circuit. In the above CE example the accuracy is limited only by the models. Given the differential gain equation we can now make estimates of second harmonic distortion and third harmonic distortion (HD2 & HD3) levels.

SPICE calculates nonlinear harmonic distortion levels using FFT processing from a number of sinewave cycles in the Transient simulation mode. This involves a huge computational effort.

Often we are after rough estimates of second harmonic distortion and third harmonic distortion from trigonometric relationships. The following uses Ed Cherry's equations [ref Cherry JAES 2000, EW Jan 1995]. It yields analytic approximations (rather than specific operating point distortions of FFT using SPICE) which are more helpful to designers showing how distortions arise and how they are related to circuit choices and temperature.

The 2[nd] and 3[rd] harmonic distortion figures to be estimated for any AC level and bias voltage using

$$HD2 = \frac{\delta G1 - \delta G2}{8} \cdot 100\% \qquad HD3 = \frac{\delta G1 + \delta G2}{24} \cdot 100\% \quad \text{where} \tag{107}$$

$$\delta G1 = \frac{G1 - G0}{G0} \qquad \delta G2 = \frac{G2 - G0}{G0} \tag{108}$$

where and G1 and G2 are the gains at the dip and peak of the AC cycle and G0 is the gain at the zero crossing. These are calculated using Equation (63) where G1 is obtained with *Vin = Vdc+Vpk* and G2 is obtained with *Vin = Vdc – Vpk* for an AC sinewave superimposed on a DC bias point. Then a plot can display HD2 and HD3 as Vdc is varied over a range op bias settings.

The model used in these equations ignores saturation of the transistor. PAK105 showed saturation can be added to limit the output voltage swing to the available supply voltage which includes limiting the collector current to reflect the Vce saturation voltage limit of a BJT. For THD estimates to be valid *a test for saturation* is necessary since but the slightest saturation will invalidate the distortion estimates.